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Quantum equilibria for macroscopic systems

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Abstract

Nash equilibria are found for some quantum games with particles with spin-$1/2$ for which two spin projections on different directions in space are measured. Examples of macroscopic games with the same equilibria are given. Mixed strategies for participants of these games are calculated using probability amplitudes according to the rules of quantum mechanics in spite of the macroscopic nature of the game and absence of Planck's constant. A possible role of quantum logical lattices for the existence of macroscopic quantum equilibria is discussed. Some examples for spin-1 cases are also considered.

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1. Introduction

It often happens that mathematical structures find natural applications somewhere out of their origination. The formalism of quantum mechanics is not an exception of this rule. Applied initially to the microworld, this formalism can be used for modelling some macroscopic interactions with an element of indeterminacy. In some papers [1–3] a connection of quantum mechanics with decision theory is discussed. Quantum game theory based on quantum theory of microparticles described by quantum physics is a well-developed theory today. Nash equilibria defined by the wavefunction or the density matrix are found for different cases, the role of entangled states is shown to be important for the possibility of new solutions of some traditional game problems like 'the prisoner's dilemma' etc [4–6].

However, one can say that the class of phenomena finding their natural explanation in terms of principles of quantum mechanics is much wider. In the pioneering book [7] written before the 'epoch of quantization of games and decisions' as well as in a more recent book [8] it was shown that the quantum mechanical formalism can be applied to the description of some macroscopic systems mainly those when the distributivity property for random events is broken (see some examples for application of quantum-like models in cognitive sciences [20, 21]). In the physics of the microworld, non-distributivity has an objective status and must be present in principle. For macroscopic systems, the non-distributivity of random events expresses some specific case of the observer's 'ignorance' different from the standard probabilistic interpretation. However, as it was shown by Birkhoff and von Neumann [9] it is breaking of distributivity which leads to the Hilbert space formulation of quantum physics and use of wavefunctions as vectors of this space playing the role of the probability amplitudes. Then it is possible due to the Luders rule to make calculations of the probabilities of definite answers for physical questions concerning properties of the quantum system. These questions are described by self-conjugate operators or in more simple cases by 'yes–no' questions, being elements of the Boolean (distributive) sublattices of the general non-Boolean lattice of properties.

That is why in our papers [10–13] we looked for macroscopic games in which the necessity of use of the wavefunction as the probability measure instead of the standard Kolmogorovian measure was motivated by breaking of the distributivity property due to special rules of the game. However, in [13] we discussed the other way of looking at macroscopic quantum games. The idea is to look at one quantum game as some unity of many classical games described as Boolean sublattices of the orthocomplemented quantum logical lattice with the special prescription for probability distributions. This prescription is defined by the Luders rule for probabilities of getting definite values for complementary properties. One can even forget about the quantum logical lattice altogether and consider the following problem. Let two players Alice and Bob play two or more parallel classical games, but the mixed strategies used by them are not independent for different games—the corresponding probabilities are defined by the Luders rule by means of some wavefunction.

Let us consider the rule for spin measurements. For spin-$1/2$ and spin-1 cases with different spin projections on some directions in space measured, the wavefunctions can be parameterized by some angle. Observables also are defined by some angle. The angles for observables are considered to be fixed. Alice and Bob have the freedom to define the wavefunctions. The average profit is calculated by the quantum rule as some expectation value of the sum of non-commuting operators in Hilbert space of the quantum spin-$1/2$ or spin-1 system. Nash equilibria are found for a special choice of wavefunctions and shall be called quantum Nash equilibria. For macroscopic cases when the macroscopic quantum game is defined as the totality of parallel games with the Luders rule for strategies, one obtains the same Nash equilibria as for corresponding spin quantum systems. This interpretation of the macroscopic quantum game is different from what was published by us in our previous publications [10–13]. In the present paper, the analytical calculation of Nash equilibria is made while previously only approximate numerical methods were used.

As to the application of our results, we look for them not in physics but in economics and social sciences where some similarities with quantum physics can occur. In our examples, we look for macroscopic imitation of only some quantum properties arising due to non-distributivity of the lattice or non-commutativity of operators leading to complementarity. We do not imitate non-locality or entanglement. The lattices considered in our paper can be embedded into some enlarged Boolean lattices (see [18]), so there is no problem with the Kochen–Specker theorem and the macroscopic objectivity.

The plan of the paper is as follows. First we consider the quantum game based on microparticles with spin-$1/2$ or spin-1. Projections of spin in different directions in space are measured. This can be called the 'Stern–Gerlach game'. There are two beams of silver ions. Alice and Bob do measurements of different spin projections using Stern–Gerlach magnets. Some payoff matrix is defined. The results of measurements are random but frequencies are defined by wavefunctions in which Alice and Bob prepared particles in their beams. Then comparing the results one of the participants (let it be Bob) pays to the other one some money. The average profit of Alice can be calculated by use of the quantum rule.

In section 3 the macroscopic quantum game as two or more parallel games played by two participants with dependence of probabilities given by the Luders rule is discussed. This dependence is parameterized by some angle considered to be fixed. Putting away the wavefunction one can formulate this dependence in terms of some equation for strategies in parallel games.

In section 4 Nash equilibria are found for the initial quantum game and macroscopic quantum game by using analytical methods. Some solutions are found for the spin-$1/2$ case.

In section 5 some examples of getting 'quantum dependence' of probabilities in macroscopic games by breaking the distributivity rule are considered. Spin-$1/2$ and spin-1 quantum systems are taken as generating corresponding quantum games.

2. The Stern–Gerlach quantum game

It is very easy to organize the quantum game using some well-known experiments with quantum microparticles. To do this, one must write some payoff matrix showing what sums of money one's partner must pay to the other depending on the results of the experiments. The advantage of the quantum games in comparison with the classical ones is the 'objective' nature of chance in it.

In classical games chance occurs due to some ignorance, and that is why it is always possible for one of the partners with more exact information to have the privilege over the other. In the quantum game based on measurements of some complementary observables, the result of the individual measurement is unpredictable in principle and only some average values can be predicted if the wavefunction is known to the participants of the game. From this point of view, quantum games like quantum cryptography with its conspiracy of the information in communication give another example of the advantages of quantum physics arising due to features considered by some (the most famous was A Einstein with his dislike of ultimate chance and belief in existence of some hidden variables) as its weakness!

Any game supposes the possibility of participants making some choices dependent on their abilities. So in quantum games participants have the freedom of preparation of the wavefunctions or density matrices for microparticles and their profit will depend on their skill. In quantum games, one has the combination of two different choices: the first choice is the manifestation of the free will of the human participant in the preparation procedure, and the second is the free choice of nature manifested in the result of measurements.

Let us consider the example of what can be called the Stern–Gerlach quantum game based on the well-known Stern–Gerlach experiment. There are two different beams of silver ions in different experimental set-ups which can be located at different places. The participants called Alice and Bob prepare their atoms in the state with some wavefunctions, so that every particle in one beam has the same wavefunction. In different beams the wavefunctions are different. Call them Ψ_A and Ψ_B. Then Alice and Bob measure using Stern–Gerlach magnets at first one projection of the spin and then the other one. Spin projections can be different for different participants but they are fixed. The only freedom for the participants is in the change of the wavefunctions. The payoff matrix can be such that if Alice obtained some definite result for one projection and Bob for the other one fixed by this matrix then Bob pays to Alice some money. However, these results can be obtained only with some probabilities. The average profit of Alice is calculated then by the rule of the quantum physics as the expectation value

Table 1. Payoff matrix of Alice.

$A\diagdown B$	**1**	**2**	**3**	**4**
1	0	0	c_3	0
2	0	0	0	c_4
3	c_1	0	0	0
4	0	c_2	0	0

of some combination of spin operators for the two particle system. If the beams are different as they are supposed in our game then there is no symmetrization of wavefunctions.

The average profit is different for different choices of wavefunctions. The aim of Alice is to get the maximal average profit. She can control her wavefunction but not that of Bob. For some choice of both partners it leads to Nash equilibrium, i.e. the choice optimal for both partners. This means that for the antagonistic game if one partner gets the maximal profit the other one has the minimal loss. An interesting feature of the quantum game is that in spite of the fact that the wavefunction is the description of the pure state the expectation value defining the average profit from the point of view of the game theory corresponds to the mixed strategy used by nature.

So at first we consider the quantum game when only two spin projections are measured. Let the payoff matrix be given by table 1. Alice measures some values of the spin projections S_z and S_θ so that to definite eigenvalues of one spin projection operator correspond two orthogonal projector operators, call them A_1 and A_3 for one projection and A_2, A_4 for the other. The same is valid for Bob. But his projectors will be called B_1, B_2, B_3, B_4.

$$A_1 + A_3 = I, \quad A_2 + A_4 = I, \qquad B_1 + B_3 = I, \quad B_2 + B_4 = I$$

The meaning of the payoff matrix for our quantum game is that if Alice gets the result of her measurement **1** and Bob gets **3** then Bob pays to Alice the sum. If Alice gets **2** and Bob **4** then he pays and so on. Alice in the result of the game gets the average profit calculated by the rule of quantum mechanics as the expectation value of the 'profit operator'

$$H = c_3 A_1 \otimes B_3 + c_1 A_3 \otimes B_1 + c_4 A_2 \otimes B_4 + c_2 A_4 \otimes B_2.$$

So her average profit becomes

$$\langle H\rangle = \langle\varphi|\langle\psi|H|\psi\rangle|\varphi\rangle = c_3 p_1 q_3 + c_1 p_3 q_1 + c_4 p_2 q_4 + c_2 p_4 q_2, \tag{1}$$

where $|\varphi\rangle$, $|\psi\rangle$ are normalized vectors of states expressing the mixed strategies of Alice and Bob, $p_i = \langle\varphi|A_i|\varphi\rangle$, $q_j = \langle\psi|B_j|\psi\rangle$ are the squares of the probability amplitudes given by the Luders rule

$$p_1 + p_3 = 1, \qquad p_2 + p_4 = 1, \qquad q_1 + q_3 = 1, \qquad q_2 + q_4 = 1. \tag{2}$$

In this game Alice gets money and Bob is paying her. So Alice is interested to get the maximal profit and Bob to pay the minimal sum. This leads to the idea of the Nash equilibrium, i.e., to the choice of such wavefunctions that the expectation value is maximal for one variable depending on the choice of Alice and minimal for the other depending on Bob's choice.

Our quantum game can be compared with the 'classical' game with the same payoff matrix but with mixed strategies. The payoff function for this game is written as

$$h = c_3\alpha_1\beta_3 + c_1\alpha_3\beta_1 + c_4\alpha_2\beta_4 + c_2\alpha_4\beta_2$$

where α_i, β_j are some Boolean variables with values 0 or 1 depending on the use of the corresponding strategy. So

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1, \qquad \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1.$$

The average profit for the unlimited repeat of the game is calculated by the classical von Neumann expression

$$\langle h\rangle = c_3 p_1 q_3 + c_1 p_3 q_1 + c_4 p_2 q_4 + c_2 p_4 q_2.$$
$$p_1 + p_2 + p_3 + p_4 = 1, \qquad q_1 + q_2 + q_3 + q_4 = 1.$$

The comparison of the expression for the quantum game average profit with the classical one shows that the quantum game can be obtained from the classical one by the 'quantization' procedure. The Boolean variables α_i, β_j are transformed into projector operators so that some of them do not commute. These projectors form the structure of the non-distributive orthomodular lattice called the quantum logical lattice.

After writing the average profit in terms of probabilities one can see that the main difference between the quantum game and the classical one is in normalization of probabilities. For the classical case the probabilities are normalized to 1, for the quantum case due to the Luders rule they are normalized to 2 if two non-commuting observables are measured. This corresponds to existence of two Boolean sublattices of the non-Boolean quantum logical lattice. Kolmogorovian probability measure can exist only on these sublattices, on non-Boolean lattice only the so-called quantum measure defined by the wavefunction can be defined. The Stern–Gerlach quantum game can be generalized for the cases when three and more non-commuting spin projections are measured. Then the average profit of Alice will be constructed as the sum of three or more expectation values of the corresponding observables which can be written as the expectation value of the profit operator being the sum of non-commuting operators. In our publication [12] the example of three observables was considered.

One can formulate the same quantum game for the spin-1 system with two or more non-commuting observables measured. For the case of two spin projections this was considered in our publication [11]. However, one cannot consider this quantum system as made from photons because in our example [11] all three values of spin-1 projection $1, 0, -1$ were considered while for photons due to the zero mass one has only two different polarization states. However, the quantum game of the 'Stern–Gerlach type' can be constructed for photons if, for example, linear and circular polarizations are measured.

For the spin-$1/2$ Stern–Gerlach quantum game if two non-commuting spin operators are measured one can consider for simplicity real two-dimensional space and take two-dimensional vectors in it as wavefunctions. Then our projectors can be defined as projectors on two vectors on the plane with some angle between them. So in this simple case considered in [10] there are two different angles: one parameterizing the wavefunction, the other the spin projections. In our game, the angle between spin projections is considered to be fixed while the angle defining the wavefunction can be varied expressing thus the freedom of participants of the game to prepare their wavefunctions in different ways.

3. Macroscopic quantum games

To look for macroscopic examples of games described by the mathematical formalism of quantum physics here we consider the simple case based on the Luders rule understood as some dependence of probability measures for different experiments.

If some macroscopic player Alice is playing two games at once using for her strategies probabilities different for different games where the difference is described just by the quantum Luders rule then this will be our quantum game. The average profit is calculated as the sum of profits in two games and it is calculated as the quantum expectation value. Nash equilibria for this combination of two games considered as one game can be found as in microscopic

Table 2. Payoff matrix of Alice in binary game.

$A \diagdown B$	**1-2**	**1-4**	**3-2**	**3-4**
1-2	0	c_4	c_3	$c_3 + c_4$
1-4	c_2	0	$c_2 + c_3$	c_3
3-2	c_1	$c_1 + c_4$	0	c_4
3-4	$c_1 + c_2$	c_1	c_2	0

quantum game by varying the angle defining the wavefunction. However, in our macroscopic case there is no necessity to use the notion of the wavefunction. In macroscopic situations quantum games occur due to special form of dependence of strategies in different classical games. This dependence can be due to some asymmetry in acts of the player simultaneously playing different classical games. For example he (she) cannot have the same frequency for acts done by the right or left hand etc. For the quantum game when three non-commuting spin observables are measured this dependence can be manifested in Heisenberg uncertainty relations for spin written in the form of some relations for frequencies in three classical games.

The Luders rule gives for the probabilities of getting definite answers for spin projection expressions

$$p_1 = \cos^2\alpha, \qquad p_2 = \cos^2(\alpha - \theta), \qquad q_1 = \cos^2\beta, \qquad q_2 = \cos^2(\beta - \tau). \tag{3}$$

Suppose Alice is playing the games on two desks: one called 'even', the other one 'odd'. The same is for Bob. The average profits for Alice in each of the parallel games are

$$\langle H\rangle_{\mathrm{odd}} = c_3 p_1 q_3 + c_1 p_3 q_1, \qquad \langle H\rangle_{\mathrm{even}} = c_4 p_2 q_4 + c_2 p_4 q_2.$$

So for the average profit in two games one obtains

$$\langle H\rangle = c_3 p_1 q_3 + c_1 p_3 q_1 + c_4 p_2 q_4 + c_2 p_4 q_2.$$

The important feature of these classical games making them different from well-known situations is the existence of 'quantum cooperation' given by formulae (3) with fixed $0 < \theta, \tau < 90°$.

This cooperation can be written in a more symmetric form as some equation for p_1, p_2. To do this, one can introduce new variables

$$\xi = -1 + p_1 + p_2, \qquad \eta = -p_1 + p_2$$

so that by use of (3) after simple trigonometric operations one obtains

$$\frac{\xi^2}{\cos^2\theta} + \frac{\eta^2}{\sin^2\theta} = 1, \tag{4}$$

i.e., the equation of the ellipse with axes defined by $\cos\theta$, $\sin\theta$. The same equation with angle τ one obtains for Bob. So the Luders rule in our case means the existence of specific 'quantum correlation'. The existence of this correlation is the new feature of our games, making possible to consider it as one macroscopic quantum game.

Let us note that the 'quantum correlation' arising due to the existence of the wavefunction and Luders rule is not the same as classical correlation. In fact, if one considers two games as one antagonistic classical game (table 2) the possible strategies can be considered as '**1-2**', '**3-2**', '**1-4**', '**3-4**'. The same is for Bob. Then introducing mixed strategies of Alice and Bob in this classical matrix game as p_{ik}, q_{ik} one has

$$p_1 = p_{12} + p_{14}, \qquad p_3 = p_{32} + p_{34}, \qquad p_2 = p_{12} + p_{32}, \qquad p_4 = p_{14} + p_{34}. \tag{5}$$

It is evident that $p_1 + p_3 = 1$, $p_2 + p_4 = 1$. Acts of Alice in two games can be independent, i.e.,

$$p_{12} = p_1 p_2, \qquad p_{32} = p_3 p_2, \qquad p_{14} = p_1 p_4, \qquad p_{34} = p_3 p_4. \tag{6}$$

But the equation for correlation can be still valid. In contrast, classical correlation means breaking of (6).

One can see the other sense of 'quantum cooperation'. Quantum cooperation means that if $\alpha = \theta$ then the 'even' game is deterministic but the 'odd' game for $\theta \neq 0$ cannot be deterministic. For $\alpha = 0$ the 'odd' game is deterministic but then the 'even' is random. This is manifestation of complementarity due to non-commutativity of the corresponding operators. Can one look for such situations in economics, politics? It seems that the answer is positive.

4. Nash equilibria

In our publication [10] we used numerical methods to look for Nash equilibria in the Stern–Gerlach spin-1/2 quantum game. Here we shall use more exact analytical methods.

Introduce the variables x_1, x_2 for Alice and y_1, y_2 for Bob as

$$x_1 = \frac{-1 + p_1 + p_2}{\cos\theta}, \qquad x_2 = \frac{-p_1 + p_2}{\sin\theta}, \qquad y_1 = \frac{-1 + q_1 + q_2}{\cos\tau}, \qquad y_2 = \frac{-q_1 + q_2}{\sin\tau}.$$

Then equations (4) for 'quantum cooperation' become

$$x_1^2 + x_2^2 = 1, \qquad y_1^2 + y_2^2 = 1.$$

So the strategy of the participant of the game is defined by a point on the unit circle and to the game situation corresponds the point on the two-dimensional torus. Back transformations from vectors to probability distributions are written as

$$2p = M_\theta x + e, \qquad 2q = M_\tau y + e, \tag{7}$$

where

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \cos\theta & \sin\theta \end{pmatrix}, \qquad e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The same formula is valid for M_τ.

Introduce notation $n = c_1 + c_3$, $m = c_2 + c_4$,

$$a = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad b = \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}, \qquad C = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}.$$

Then the payoff function can be written in matrix form as

$$\langle H \rangle = -\langle p, Cq \rangle + \langle b, p \rangle + \langle a, q \rangle.$$

After change of variables (7) one obtains

$$\begin{aligned} 4\langle H \rangle &= -\langle M_\theta x + e, C(M_\tau y + e) \rangle + 2\langle b, M_\theta x + e \rangle + 2\langle a, M_\tau y + e \rangle \\ &= -\left\langle x, M_\theta^\dagger C M_\tau y \right\rangle + 2\langle b, M_\theta x \rangle + 2\langle a, M_\tau y \rangle \\ &\quad - \langle M_\theta x, Ce \rangle - \langle M_\tau y, Ce \rangle - \langle e, Ce \rangle + 2\langle a, e \rangle + 2\langle b, e \rangle. \end{aligned}$$

This can be written as

$$4\langle H \rangle = h(x, y) + \langle e, Ce \rangle, \tag{8}$$

with

$$h(x, y) = -\left\langle x, M_\theta^\dagger C M_\tau y \right\rangle + \left\langle x, M_\theta^\dagger \omega \right\rangle - \left\langle M_\tau^\dagger \omega, y \right\rangle, \tag{9}$$

where $\omega = b - a$. The vector variables x, y satisfy the limitations

$$|x| = 1, \qquad |y| = 1. \tag{10}$$

So one has the problem of Nash equilibria of h on torus. To solve the problem first prove some general properties:

Proposition 1. *$x_{\min}$ is the point of minimum and $x_{\max}$ is the point of maximum of a function $f(x) = \langle k, x\rangle + b$ on the circle if and only if for some non-negative λ the equalities $k = \lambda x_{\max}$, $k = -\lambda x_{\min}$ are valid.*

The proposition is proved if one pays attention to the fact that the only variable is the angle in the scalar product.

Denote

$$A = M_\theta^\dagger C M_\tau, \qquad u = M_\theta^\dagger \omega, \qquad v = M_\tau^\dagger \omega, \tag{11}$$

then

$$h(x, y) = -\langle x, Ay\rangle + \langle x, u\rangle - \langle v, y\rangle. \tag{12}$$

Using proposition 1 one obtains

Proposition 2. *The point (x, y) is the Nash equilibrium for the function h if and only if non-negative numbers λ and μ exist satisfying equalities*

$$-Ay + u = \lambda x, \qquad A^\dagger x + v = \mu y. \tag{13}$$

The proposition is proved solving the problem of the conditional extremum problem with λ, μ being the Lagrange multipliers.

Proposition 3. *If the point (x, y) is the Nash equilibrium for the game on torus with the payoff function $h(x, y) = -\langle x, Ay\rangle + \langle x, u\rangle - \langle v, y\rangle$ then for some non-negative λ, μ the following equations are valid:*

$$(AA^\dagger + \lambda\mu I)x = \mu u - Av, \qquad (A^\dagger A + \lambda\mu I)y = \lambda v + A^\dagger u. \tag{14}$$

To obtain (14) multiply the first equation in (13) by μ, the second by λ and then act by the operator $A^\dagger$ on the first equation and by A on the second. It is also easy to see that matrices $A^\dagger A$ and $AA^\dagger$ are positive and isospectral. From these general properties it is easy to get the following criterion.

Proposition 4. *If $a = b$ then the Nash equilibrium is impossible.*

In this case $\omega = 0$. From (11) it follows that $u = v = 0$ so by (14) for some $\lambda, \mu \geqslant 0$ one has $(AA^\dagger + \lambda\mu I)x = 0$, $A^\dagger A + \lambda\mu I)y = 0$. But for non-negative λ, μ matrices $AA^\dagger + \lambda\mu I$, $A^\dagger A + \lambda\mu I$ are positive defined, so $x = y = 0$ which is impossible.

This result was also obtained by us previously [10]. It is interesting that for the classical game the Nash equilibrium always exists in mixed strategies while for the quantum game it is not always so. Examine some special cases of existence of Nash equilibria for the quantum game.

Let ω be not equal to zero. Then $u = M_\theta^\dagger \omega$, $v = M_\tau^\dagger \omega$ are also not zero.

Proposition 5. *Operators $AA^\dagger$ $A^\dagger A$ have the form*

$$AA^\dagger = \begin{pmatrix} (m^2 + 2mn\cos 2\tau + n^2)\cos^2\theta & (m^2 - n^2)\sin\theta\cos\theta \\ (m^2 - n^2)\sin\theta\cos\theta & (m^2 - 2mn\cos 2\tau + n^2)\sin^2\theta \end{pmatrix}$$

$$A^\dagger A = \begin{pmatrix} (m^2 + 2mn\cos 2\theta + n^2)\cos^2\tau & (m^2 - n^2)\sin\tau\cos\tau \\ (m^2 - n^2)\sin\tau\cos\tau & (m^2 - 2mn\cos 2\theta + n^2)\cos^2\tau \end{pmatrix}.$$

If $m = n$ and $\theta = \tau = 45^\circ$ then operators $AA^\dagger$ and $A^\dagger A$ are proportional to the unit: $AA^\dagger = A^\dagger A = m^2 I$.

Proposition 6. *If $m = n$, $\theta = \tau = 45^\circ$ and $n^2 \leqslant \omega_1^2 + \omega_2^2$ then there exists one point of Nash equilibrium (x, y), such that*

$$x = y = \frac{1}{\sqrt{2\left(\omega_1^2 + \omega_2^2\right)}} \begin{pmatrix} \omega_2 - \omega_1 \\ \omega_2 + \omega_1 \end{pmatrix}.$$

The probabilities are equal to

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{2|\omega|} \begin{pmatrix} c_3 - c_1 + |\omega| \\ c_4 - c_2 + |\omega| \end{pmatrix}, \qquad \textit{with} \qquad |\omega| = \sqrt{\omega_1^2 + \omega_2^2}.$$

Then the optimal value of the profit is $\langle H \rangle = n/4$.

This is proved by use of (13), (14) for $u = v$ and taking $\mu = 3m$, $\lambda = m$. From (13) one obtains optimal strategies as $x = y = u/2m$. Then

$$u = M^\dagger \omega = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_2 - \omega_1 \\ \omega_2 + \omega_1 \end{pmatrix}.$$

Normalizing it and going to variables (7) one obtains our result.

Examples ($m = n, \theta = \tau = 45^\circ$)**.**
For $c_1 = 2, c_2 = 1, c_3 = 8, c_4 = 9$ the optimal strategies of Alice and Bob are $p_1 = q_1 = 0.8$, $p_2 = q_2 = 0.9$, $p_3 = q_3 = 0.2$, $p_4 = q_4 = 0.1$.
For $c_1 = 8, c_2 = 9, c_3 = 2, c_4 = 1$ the optimal strategies of Alice and Bob are $p_1 = q_1 = 0.2$, $p_2 = q_2 = 0.1$, $p_3 = q_3 = 0.8$, $p_4 = q_4 = 0.9$.
For $c_1 = 1, c_2 = 2, c_3 = 9, c_4 = 8$ the optimal strategies of Alice and Bob are $p_1 = q_1 = 0.9$, $p_2 = q_2 = 0.8$, $p_3 = q_3 = 0.1$, $p_4 = q_4 = 0.2$.

These results are in agreement with the properties of quantum logic (see the next section). In fact, the second set of payoffs is obtained from the first by permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

expressing the automorphism of the lattice changing the elements on their orthocomplements.

The third set of payoffs is obtained from the first one by the automorphism of the lattice

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

The average profit for all three cases will be the same and is equal to $\langle H \rangle = 2.5$. This can be compared with the average profit for the Nash equilibrium for the classical game (without quantum cooperation)

$$h = \left(c_1^{-1} + c_3^{-1}\right)^{-1} + \left(c_2^{-1} + c_4^{-1}\right)^{-1} = 2.5.$$

So in this case it has the same value for the quantum case and the classical one. However, such a coincidence is not necessary. This can be shown by the following example. For $c_1 = 1, c_2 = 9, c_3 = 10, c_4 = 2$ the optimal strategies of Alice and Bob are

$$p_1 = q_1 = \frac{130 + 9\sqrt{130}}{260} \approx 0.895, \qquad p_2 = q_2 = \frac{130 - 7\sqrt{130}}{260} \approx 0.193.$$

The optimal profit in the classical game is smaller than in the quantum one: $\langle h \rangle = 28/11$, $\langle H \rangle = 11/4$.

Now let us consider another special case which can be called the case of 'eigenequilibria'. Suppose matrices $A^\dagger$ and $A^\dagger A$ are not multiples of the identity matrix. Then one can prove the following propositions.

Proposition 7. *Let u be the eigenvector of the operator $AA^\dagger$, and v be the eigenvector of the operator $A^\dagger A$ and their eigenvalues are equal. Then for some s, t one has*

$$A^\dagger u = sv, \qquad Av = tu.$$

Here $s \cdot t$ is the eigenvalue of the operators mentioned.

From this one can obtain the following.

Proposition 8. *Let u be the eigenvector of the operator $AA^\dagger$, and v be the eigenvector of the operator $A^\dagger A$ and the eigenvalues are equal. Then the strategies $x = u/|u|$, $y = v/|v|$ satisfy the necessary conditions of equilibrium for some λ, μ and one has*

$$-Ay + u = \lambda x, \qquad A^\dagger x + v = \mu y.$$

Proposition 9. *The relations*

$$A^\dagger u = sv, \qquad Av = tu$$

are valid if and only if vector ω is the general eigenvector of matrices $CM_\theta M_\theta^\dagger$, $CM_\tau M_\tau^\dagger$.

Proposition 10. *Let ω be the general eigenvector of operators $M_\theta M_\theta^\dagger$ and $M_\tau M_\tau^\dagger$, and s is the eigenvalue of the operator $CM_\theta M_\theta^\dagger$. The pair of strategies $x = u/|u|$, $y = v/|v|$ defines the Nash equilibrium if and only if $s < |u|$.*

Proposition 11. *Let both components of ω be different from zero and there are inequalities*

$$m^2\omega_1^4 - n^2\omega_2^4 > m^2n^2(\omega_1^2 - \omega_2^2) \tag{15}$$

$$(\omega_1^2 - \omega_2^2)(m^2\omega_1^2 - n^2\omega_2^2) > 0. \tag{16}$$

Then the condition

$$\cos 2\tau = \frac{(n-m)\omega_1\omega_2}{m\omega_1^2 - n\omega_2^2} = \cos 2\theta \tag{17}$$

is necessary and sufficient for the existence of equilibrium.

The remark. Eigenequilibrium is possible only for equal angle parameters of the game.

Proposition 12. *For conditions (15)–(17) the probabilities are equal to*

$$p = q = \frac{1}{2}\left(\sqrt{\frac{\omega_1^2 - \omega_2^2}{m^2\omega_1^4 - n^2\omega_2^4}}\begin{pmatrix} m\omega_1 \\ n\omega_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

Example. Let $\theta = \tau = 45^\circ$ and $c_1 = 0$, $c_2 = 1$, $c_3 = 1$, $c_4 = 1$ then the optimal strategies of Alice and Bob are

$$p_1 = q_1 = 1, \qquad p_2 = q_2 = 0.5, \qquad p_3 = q_3 = 0, \qquad p_4 = q_4 = 0.5.$$

In this case, the profit has the same value for the quantum case and the classical one: $\langle H\rangle = \langle h\rangle = 0.5$.

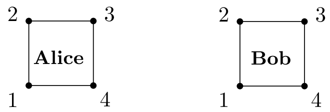

Figure 1. Binary game of Alice and Bob.

5. Macroscopic quantum games and the quantum logic

In the previous section, we discussed the idea of the macroscopic quantum game as the system of classical games with a special condition on the strategies. However, we did not consider the origin of this condition, i.e. in what situations such conditions necessarily arise.

Here we give some examples when this is so. These examples are based on the connection first mentioned by D Finkelstein [14] and then developed in the works of A A Grib and R R Zapatrin [15] between quantum logical lattices and graphs. These examples are taken from publications [10–13]. Considering the idea of existence of macroscopic situations described by the formalism of quantum physics one must also mention the publications of D Aerts [16].

It was J von Neumann who in his paper with G Birkhoff [9] first mentioned that the structure of properties of the quantum system for the simple spin-$1/2$ system is the structure of the orthocomplemented non-distributive lattice. Non-distributivity leads to non-commutativity of projector operators representing the abstract lattice. These lattices were called quantum logical lattices or simply 'quantum logics'. Later the ideas of von Neumann were developed by Jauch and Piron [17] for more general cases and now form the basis of the axiomatic of the quantum physics. Non-distributivity means that if there are properties A, B, C then using notation $\wedge$ for 'and', notation $\vee$ for 'or',

$$(A \vee B) \wedge C \neq (A \wedge C) \vee (B \wedge C).$$

Breaking of the distributivity means that operations $\wedge, \vee$ cannot be understood as usual conjunctions and disjunctions of the set theory. The structure of the non-distributive lattice is not Boolean and one cannot define on such structures the standard Kolmogorovian probability measure. In addition to quantum mechanics non-Boolean lattices arise for topologies (see [19]) so that if topologies are considered as random one also cannot define for them the standard probability measure. However, for quantum mechanical examples one can define the probability amplitude or 'quantum probability measure' represented by some vector in Hilbert space.

In [10] the game called 'wise Alice' was considered. Let Alice and Bob play the following game. Alice and Bob have two quadrangles, one for Alice and one for Bob (figure 1). Let Bob put some ball to the vertex of the quadrangle and Alice must guess to what vertex he did that. She asks Bob the question: 'Did you put it into **1**?' The rule of the game is such that Bob always answers 'yes' if he is in **1**, but he gives the same answer if the ball was in **2** or **4**, i.e. in the vertices connected with **1** by one arc. However, it is prohibited for Bob to move by two steps from **3** to **1** and so if he is in **3** then he always answers her question as 'no'. The same rule is valid for any vertex. Alice, however, knows this property of 'accommodation' of Bob to her questions. This leads to specific logic of Alice: she pays no attention to affirmative answers of Bob and notes only his negative answers. Then it is easy to see that different positions of the ball of Bob will be described due to negative logic as disjunctive, i.e. $\mathbf{1} \wedge \mathbf{2} = \mathbf{1} \wedge \mathbf{3} = \mathbf{1} \wedge \mathbf{4} = \mathbf{3} \wedge \mathbf{4} = \mathbf{2} \wedge \mathbf{3} = \mathbf{2} \wedge \mathbf{4} = \varnothing$ but the disjunction is now not unique $\mathbf{1} \vee \mathbf{2} = \mathbb{I}$. Here $\mathbb{I}$ means 'always true', and $\varnothing$ means 'false'. From the structure of the graph and the rules of the game it is easy to see that due to Bob's 'accommodation'

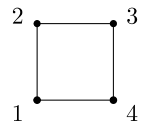

Figure 2. Graph of Alice and Bob.

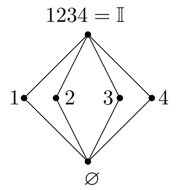

Figure 3. Lattice of Alice and Bob.

there is no difference for Alice between the situation $\mathbf{1} \vee \mathbf{2}$ and $\mathbf{1} \vee \mathbf{2} \vee \mathbf{3} \vee \mathbf{4}$. In figure 3 we show the connection of the graph (the upper drawing) and the quantum logical lattice (the lower drawing). This lattice is a well-known lattice for Stern–Gerlach experiment when two different spin projections are measured. Lines going 'up' intersect at $\vee$ ('or'), lines going down intersect at $\wedge$ ('and'). The lower drawing is called the Hasse diagram [17].

However, to simulate the Stern–Gerlach quantum game considered in section 2 of this paper one must make the game symmetric for both partners. This means that the same rule is valid for Bob guessing to what vertex of her quadrangle Alice put her ball. So Bob also comes to the same quantum logical lattice. Putting questions to one another Alice and Bob obtain some truly guessed numbers due to negative answers positions of the balls and neglecting all 'yes' answers. Let these numbers, for Alice, be N_1, N_3 and N_2, N_4 for opposite vertices of the graph. Similar numbers are obtained by Bob. To transform these numbers into probabilities

$$\frac{N_1}{N_1 + N_3}, \qquad \frac{N_3}{N_1 + N_3}, \qquad \frac{N_2}{N_2 + N_4}, \qquad \frac{N_4}{N_2 + N_4}$$

as it is in the quantum Stern–Gerlach game one can do the following. Let the game consist of two parts as it was proposed in [17] preparation and measurement. Defining the numbers N means preparation. The second part, measurement, corresponds to the changed situation: Alice and Bob cannot accommodate to one another and now in N_1 cases Bob will be in **1**, in N_3 cases in **3** etc but Alice every time does not know exactly if he is in **1** or **3**.

Instead of one game with quadrangle there are two games with two diagonals of the quadrangle. The strategies of Alice are defined by the probabilities obtained from the first stage due to her knowledge of the numbers N. These probabilities due to the properties of the quantum logical lattice satisfy limitations defined by the wavefunction for the spin-1/2 system. The same rule is valid for Bob and he also plays two games with strategies defined by numbers obtained in the first part. The payoff is made according to the payoff matrix defined in section 2 and the average profit of one of the partners is calculated according to the quantum rule.

The necessity of going to the second part, measurement, is motivated by the fact that it is only for Boolean sublattices of the non-Boolean lattice that one can define probabilities. The difference of our macroscopic non-Boolean game from the microscopic Stern–Gerlach game is due to the fact that in the macroscopic case Alice and Bob necessarily put their balls into position defined by the question of the partner while in the microscopic case there is

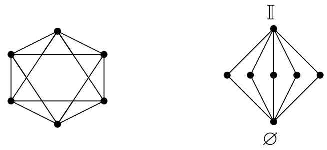

Figure 4. Graph and Hasse diagram for spin-1/2 system with three different spin projections measured.

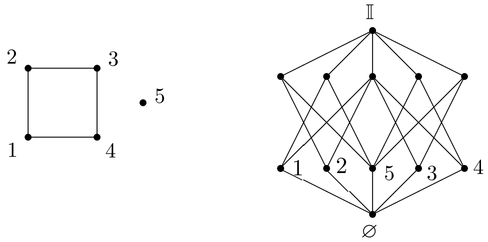

Figure 5. Graph and Hasse diagram for spin-1 system with two different spin projections measured.

indeterminism, so that there is no such necessity. However, this freedom of choice is simulated in the macroscopic game by the freedom of choice of the players in the second part to put their balls to any vertices with prescribed probabilities. The other difference is that the abstract quantum logical lattice which is the same for the microscopic and macroscopic cases has many different representations in terms of projectors. This means that the angle between projectors is not fixed by the lattice and can be any. In the microscopic Stern–Gerlach experiment, this angle is chosen by the will of the experimentalist choosing the direction of the magnetic field in his magnet.

In the macroscopic case, the angle is fixed by the ratio of probabilities for different games on the second stage. Connection between quantum logical lattices and graphs leads to the possibility of certain classification of macroscopic quantum games. For example, one can consider three classical games with strategies defined by the probabilities satisfying limitations due to the existence of the wavefunction, i.e. Stern–Gerlach experiment with three different spin projections for the spin-1/2 system being measured. The graph and the quantum logical lattice are shown in figure 4. There is the possibility of macroscopic imitation of the Stern–Gerlach experiment with spin-1 system. If two non-commuting spin projection operators are measured then the graph and the lattice are shown in figure 5. The payoff matrices and the average profit for the cases in figure 5 were considered in [11]. What is the sense of Nash equilibria for macroscopic quantum games based on quantum logic? The angle between two ‘projections’ is defined by the ratio of probabilities in two classical games corresponding to Boolean sublattices of the non-Boolean lattice. Nash equilibria for the fixed angle for projections correspond to some ‘patterns’ of stability for players. One of the partners receives the maximal profit and the other has the minimal loss in this situation so the partners can come to mutual agreement on their behaviour after experiencing many games of this type. This can have meaning for some economical situations.

One can construct some generalization of the macroscopic quantum game based on the use of quantum logic. It is possible to change the second part of the game so that only the angles between observables are defined in the first part for Alice and Bob. They have the possibility of choosing any wavefunction which means the angles defining the probability amplitudes. This will correspond exactly to the problem for Nash equilibria considered in section 4 of this paper. Some 'quantum casino' can be organized following this rule.

6. Conclusion

Macroscopic quantum games can be considered as some collections of classical games with special limitations on the strategies. For example, in section 4 it was shown that for the special case of the Stern–Gerlach game with two different $1/2$ spin projections being measured the game can be formulated as some game on the two-dimensional torus. The mathematical analysis shows that there can be three different situations for the Nash equilibrium in this case in comparison with the ordinary classical game with the same payoff matrix. There can be the case when no Nash equilibrium exists for the quantum game. There can be the case when the profit in the quantum game is equal to that in the classical game. There can be the case when it is larger than that in the classical game. The difference is defined by the structure of the payoff matrix.

Limitations on strategies form new features of the quantum games. In economics, they can be considered as some external to the payoff matrix conditions on strategies motivated by the interests of one or may be of both of the partners forming some 'quantum cooperation' as we called it. This is especially so if the profit is larger than in the classical case. There is some resemblance of our game to the so-called Parrondo's game [5, 22]. In Parrondo's game one gets the larger profit in the case when two different games are played together with special coupling between them (the player's capital etc) but not separately. However, this coupling does not have the form of the 'quantum cooperation' as it is in our paper.

Macroscopic quantum games with special Nash equilibria in them can occur if non-Boolean quantum logical lattices arise. This can be in the cases when the behaviour of partners is such that some 'accommodation' (or 'deceiving') of one player by the other takes place. Then if at some stage this accommodation becomes impossible but the behaviour is defined by the previous stage through limitations on strategies one can come to the situation of some Stern–Gerlach quantum game. Nash equilibria can be calculated as it was done in section 4. They form some patterns of behaviour which can be considered as preferable for partners. The simulation of the microscopic Stern–Gerlach quantum game can be made if in some 'quantum casino' the game is organized in two stages so that on the first stage quantum logic arising due to breaking of the distributivity rule leads to fixation of the angles and limitations on possible strategies used on the second stage consisting as combination of classical games. Nash equilibria for quantum games can be chosen by the players as preferable to them.

Other examples of macroscopic situations described by non-distributive lattices considered in our paper can be found in the book of K Svozil [18]. One of them is known as the 'firefly-in-a-box' example, the other is the Wright's generalized urn model. In both cases, there is complementarity arising due to different conditions of observation of the object which cannot be realized simultaneously. One can imagine macroscopic quantum games based on these examples.

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